$y_1 = x_1, y_2 = x_2 - \alpha(x_1)$. Here $x_2 = \alpha(x_1)$ is the equation of the boundary near the point x°. The nature of the reasoning performed in this case is analoguous to that presented above.

REFERENCES

1. FICHERA G., Existence Theorems in Elasticity Theory /Russian translation/, Mir, Moscow, 1974.
2. KRAVCHUK A.S., On the Hertz problem for linearly and non-linearly elastic bodies of finite size, PMM, 41, 2, 1977.
3. KHLUDNEV A.M., On the contact problem of a linear elastic body with elastic and rigid bodies (a variational approach), PMM, 47, 6, 1983.
4. CAFFARELLI L.A. and FRIEDMAN A., The obstacle problem for the biharmonic operator. Ann. Scuola Norm., Sup. Pisa, ser. 4, 6, 1979.
5. KHLUDNEV A.M., A variational approach to the contact problem of a shallow shell and a rigid body. Partial Differential Equations (Trudy seminara S.L. Soboleva, 2), Izd. Inst. Matematiki Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1981.
6. LIONS J.-L and MAGENES E., Inhomogeneous Boundary Value Problems and Their Application /Russian translation/, Mir, Moscow, 1971.
7. JOHN O. and NAUMANN J., On regularity of variational solutions of the von Kárman equations. Math. Nachr., 71, 1976.
8. TEMAM R., Navier-Stokes Equations. Theory and Numerical Analysis /Russian translation/, Mir, Moscow, 1981.
9. DUVAU G. and LIONS J.-L, Inequalities in Mechanics and Physics /Russian translation/, Nauka, Moscow, 1980.

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MULTIPLE EIGENVALUES IN OPTIMIZATION PROBLEMS*

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The problem of maximizing the minimum eigenvalue of a selfadjoint matrix operator is considered. The case when the optimum eigenvalue is multiple, i.e. the problem of optimization is discontinuous, is investigated. This problem has interesting applications in the optimum design of constructions /1-6/. The necessary conditions for a local maximum of the eigenvalue of arbitrary multiplicity p with an **isoperimetric** limit are obtained. The paper generalizes the results obtained in /7, 8/ for the single and double case.

Consider the eigenvalue problem

$$A[h]u = \lambda B[h]u \tag{1}$$

Here $A[h]$ and $B[h]$ are positive-definite symmetric $m \times m$ matrices with coefficients $a_{ij}(h)$ and $b_{ij}(h)$, which depends continuously on the components of the vector of the parameters h of dimensions n, u is an eigenvector of dimensions m, and λ is an eigenvalue.

Problem (1) has a complete system of eigenvectors u^i $(i = 1, 2, \ldots, m)$ and a sequence of eigenvalues λ_i $(i = 1, 2, \ldots, m)$ corresponding to this system; we will assume that the orthogonality condition is satisfied

$$(B[h]u^i, u^j) = \delta_{ij} \tag{2}$$

where δ_{ij} is the Kronecker delta. Here and henceforth the parenthesis denote the scalar product of vectors.

We will formulate the optimization problem as follows: it is required to obtain the vector of the parameters $h = (h_1, h_2, \ldots, h_n)$ for which the minimum eigenvalue λ_1 of problem (1) reaches a maximum value under the conditions

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$$F(h) = 0 \tag{3}$$

where $F(h)$ is a continuously differentiable scalar function of a vector argument.

Suppose the p-dimensional minimum eigenvalue $\lambda_1 = \lambda_2 = \ldots = \lambda_p < \lambda_{p+1} \leqslant \lambda_{p+2} \leqslant \ldots \leqslant \lambda_m$, $1 < p \leqslant m$ corresponds to a vector h, which satisfies condition (3). We will give the vector h an increment in the form of the vector εk, $\|k\| = 1$, $k = (k_1, k_2, \ldots, k_n)$, where ε is a small positive number. It follows from (3) that the vector k satisfies the condition

$$(f^\circ, k) = 0, \; f^\circ = \nabla F \tag{4}$$

As a result of the perturbation of the vector of the parameters, the multiple eigenvalue λ_1, and the eigenvectors $u^1, u^2, \ldots, u^p$ obtain increments which have the form /9/

$$\lambda = \lambda_1 + \varepsilon\mu + \varepsilon^2\eta + o(\varepsilon^2), \; u = u^\circ + \varepsilon v^1 + \varepsilon^2 v^2 + o(\varepsilon^2), \tag{5}$$
$$u^\circ = \gamma_1 u^1 + \gamma_2 u^2 + \ldots + \gamma_p u^p$$

where u° is a linear combination of the eigenvectors u^i $(i = 1, 2, \ldots, p)$. The coefficients γ_i $(i = 1, 2, \ldots, p)$ remain to be determined from the equations of the method of perturbations.

Substituting the expansions (5) into (1) we obtain, to a first approximation,

$$Cu^\circ + Av^1 - \lambda_1 Bv^1 = \mu Bu^\circ \tag{6}$$

where C is a matrix with the coefficients $c_{ij} = (\nabla a_{ij}, k) - \lambda_1 (\nabla b_{ij}, k)$, $\nabla = (\partial/\partial h_1, \partial/\partial h_2, \ldots, \partial/\partial h_n)$. Multiplying (6) scalarly by u^i $(i = 1, 2, \ldots, p)$ we obtain a system of linear equations in the constants γ_i

$$\sum_{j=1}^{p} (\alpha_{ij} - \mu\delta_{ij}) \gamma_j = 0, \quad i = 1, 2, \ldots, p; \quad \alpha_{ij} = (Cu^i, u^j) \tag{7}$$

For convenience we will introduce the vectors f^{ij} of dimensions n

$$f^{ij} = \sum_{s,t=1}^{m} u_s^i u_t^j (\nabla a_{st} - \lambda_1 \nabla b_{st}) \tag{8}$$

where u_s^i, u_t^j are the components of the eigenvectors u^i, u^j. Note that $f^{ij} = f^{ji}$, in view of the symmetry of a_{ij}, b_{ij}. Taking the notation (8) into account, the coefficients α_{ij} from (7) can be written in the form $\alpha_{ij} = (f^{ij}, k)$.

Equating the determinant of the set of Eqs.(7) to zero, we obtain an equation for determining μ

$$\det \| (f^{ij}, k) - \mu\delta_{ij} \| = 0 \tag{9}$$

Hence, knowing the eigenvalue λ_1 and the eigenvectors u^i, $i = 1, 2, \ldots, p$ corresponding to this eigenvalue, we can calculate the vectors f^{ij} using (8), and from the vector of variation k from (9) we can calculate the variations $\mu_1, \mu_2, \ldots, \mu_p$ of the p-tuple eigenvalue λ_1.

Assertion 1. If the vectors f°, f^{ij}, $ij = 1, 2, \ldots, p$; $j \geqslant i$ (of all $1/2\, p(p+1) + 1$) are linearly independent, an improved variation k exists for which $\mu_i > 0$, $i = 1, 2, \ldots, p$.

Note that the linear independence of these vectors is only possible when $n \geqslant 1/2\, p(p+1) + 1$.

Proof. Consider the system of linear equations in the components of the vector of variations $k_1, k_2, \ldots, k_n$

$$(f^\circ, k) = 0; \; (f^{ij}, k) = \delta_{ij}\nu_i^\circ, \; ij = 1, 2, \ldots, p; \; j \geqslant i \tag{10}$$

where ν_i° are specified positive constants. If the vectors f°, f^{ij}, $ij = 1, 2, \ldots, p$; $j \geqslant i$ are linearly independent, the solution (10) exists for any ν_i°, in particular when $\nu_i^\circ > 0$. Suppose the vector k is a solution of system (10); we will normalize this vector $\tilde{k} = k/\|k\|$. Then we obtain from Eqs.(9) and (10) $\mu_i = (f^{ii}, \tilde{k}) = \nu_i^\circ/\|k\| > 0$ $(i = 1, 2, \ldots, p)$, which also proves the possibility of constructing the improved variation k, for which $\mu_i > 0$ $(i = 1, 2, \ldots, p)$.

When $n < 1/2\, p(p+1) + 1$, the vectors f°, f^{ij} are always linearly dependent, and hence the improved variation cannot exist.

In the case when the vectors f°, f^{ij} $(ij = 1, 2, \ldots, p; j \geqslant i)$ are linearly dependent, we will separate from them linearly independent vectors $f^\circ, f^1, f^2, \ldots, f^{r-1}$, where r is the rank of the matrix consisting of the vectors f°, f^{ij}, $r \leqslant 1/2\, p(p+1)$, and we will expand the remaining vectors f^{ij} in terms of them

$$f^{ij} = \xi_0^{ij} f^\circ + \xi_1^{ij} f^1 + \ldots + \xi_{r-1}^{ij} f^{r-1} \tag{11}$$

Note that the coefficients of the expansion are symmetrical: $\xi_t^{ij} = \xi_t^{ji}$ in view of the fact $f^{ij} = f^{ji}$. Taking (4) and (11) into account, the coefficients α_{ij} of the matrix can be represented in the form

$$\alpha_{ij} = (f^{ij}, k) = \sum_{t=1}^{r-1} \xi_t^{ij} l_t; \quad l_t = (f^t, k) \tag{12}$$

Assertion 2. If the vector of the parameter h which satisfy condition (3) make the minimum eigenvalue λ_1 with multiplicity p a maximum, it is necessary that:

1) the vectors f°, f^{ij}, $i, j = 1, 2, \ldots, p$; $j \geqslant i$ must be linearly dependent;

2) the set of vectors $l = (l_1, l_2, \ldots, l_{r-1})$, defined by the following conditions: a) $D_1 D_3 > 0$,

$D_3D_5 > 0, \ldots, D_{p-3}D_{p-1} > 0$; $D_2 > 0$, $D_4 > 0$, $D_6 > 0, \ldots, D_p > 0$, if p is even, or b) $D_1D_3 > 0$, $D_3D_5 > 0, \ldots, D_{p-2}D_p > 0$; $D_2 > 0$, $D_4 > 0, \ldots, D_{p-1} > 0$, if p is odd, must be empty.

Here $D_1, D_2, \ldots, D_p$ are the principal minors of the matrix with coefficients α_{ij}, $ij = 1, 2, \ldots, p$.

Proof. The necessity of condition 1) follows from Assertion 1. Condition 2) denotes that the characteristic polynomial (9) will not have roots μ_i of fixed sign. The conditions on the principal minors a) or b) follow from the well-known necessary and sufficient conditions for symmetrical matrices to be positive- and negative-definite /10/.

Note that a change in the sign of the variation k changes the signs of the first variations μ_i $(i = 1, 2, \ldots, p)$. Hence, the lack of roots μ_i of fixed sign is the necessary condition for a maximum of λ_1.

When condition (12) are satisfied in the $(r-1)$-dimensional space of the vectors $l = (l_1, l_2, \ldots, l_{r-1})$ inequalities a) or b) define regions the intersection of which must be empty. This imposes conditions on the coefficients of the expansions ξ_t^{ij} from (11). These conditions are quite simple in the double case /7, 8/.

Note. The problem of maximizing the minimum eigenvalue, taking into account the multiplicity in a continuous formulation, was considered in /11/, where the conditions of linear dependence of the form (11) were given in the following form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\Omega_{ij} = k^2 \tag{13}$$

Here Ω_{ij} are functions which play the role of the vectors f^{ij} in the notation used in this paper, and k^2, γ_{ij} are the coefficients of the expansions. However, the derivation of this condition is incorrect. In particular, it follows from condition (34) of /11/, instead of (35) (condition (13)) that $\Omega_{ij} = \text{const}$ $(i, j = 1, 2, \ldots, n)$.

After this paper went to press, paper /12/ was published. This was devoted to problems of optimizing eigenvalues taking their multiplicity into account when there are limitations in the form of the inequalities $g_i(x) \leqslant 0$, $i = 1, 2, \ldots, q$. This fact changes the nature of the optimization problem. To take these limitations into account, a theorem on the incompatibility of linear inequalities was used in /12/. However, this paper contains a number of serious errors.

Example. Consider the case when the matrix A is diagonal, and has p-tuple eigenvalues for the vector of the parameters h, while the matrix B is a unit matrix. In this case $\lambda_i = a_{ii}(h) = \lambda_1$, $i = 1, 2, \ldots, p$; $a_{ii}(h) > \lambda_1$, $p < i \leqslant m$.

We will obtain the first corrections μ of the p-tuple eigenvalue λ_1 from Eq. (9): $\mu_i = (f^{ii}, k)$ $(i = 1, 2, \ldots, p)$. The vectors f^{ii} in this case are gradients of the eigenvalues λ_i, $f^{ii} = \nabla a_{ii}(h)$. We will assume that they are linearly independent.

The necessary condition for the optimality of (11) can be written in the form

$$\zeta_0 f^0 + \sum_{i=1}^{p}\zeta_i f^{ii} = 0 \tag{14}$$

We will obtain the conditions imposed on the coefficients ζ_i. From conditions a) or b) of Assertion 2 we have the following inequalities:

$$(f^{11}, k)(f^{22}, k) > 0, \quad (f^{22}, k)(f^{33}, k) > 0, \ldots, (f^{p-1,p-1}, k)(f^{pp}, k) > 0 \tag{15}$$

From (14) we express, for example, the vector f^{11} in terms of f^0, f^{ii} $(i = 2, 3, \ldots, p)$ and we substitute it into (15). As a result we obtain

$$\begin{gathered} -\frac{\zeta_2}{\zeta_1}(f^{22}, k)^2 - \frac{\zeta_3}{\zeta_1}(f^{22}, k)(f^{33}, k) - \ldots - \frac{\zeta_p}{\zeta_1}(f^{22}, k). \\ (f^{pp}, k) > 0; \quad (f^{ii}, k)(f^{i+1,i+1}, k) > 0, \quad i = 2, 3, \ldots, p-1 \end{gathered} \tag{16}$$

The set of vectors k, which satisfies conditions (16) and (4), must be empty. Hence, we obviously have the inequalities

$$\zeta_i/\zeta_1 \geqslant 0, \quad i = 2, 3, \ldots, p \tag{17}$$

i.e. the conditions for the coefficients ζ_i $(i = 1, 2, \ldots, p)$ to be of fixed sign. Conditions (14) and (17) are the usual conditions for a maximin /13/.

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REFERENCES

1. OLHOFF N. and RASMUSSEN S.H., On single and bimodal optimum buckling loads of clamped columns. Intern. J. Solids and Struct., 13, 7, 1977.
2. ARMAN ZH. L.P., LUR'YE K.A. and CHERKAYEV A.V., The solution of problems of optimizing eigenvalues that arise when designing elastic constructions, Izv. Akad. Nauk SSSR, MTT, 5, 1978.
3. MEDVEDEV N.G., Some spectral features of optimal problems of the stability of shells of variable thickness, Dokl. Akad. Nauk UkrSSR, Ser. A, 9, 1980.
4. CHOI K.K. and HAUG E.J., Optimization of structures with repeated eigenvalues. In: Optimization of Distributed Parameter Structures. Alphen aan den Rijn: Sijthoff and Noordhoff, 1981.
5. BLACHUT J. and GAJEWSKI A., On unimodal and bimodal optimal design of funicular arches. Intern. J. Solids and Struct., 17, 7, 1981.
6. SEIRANYAN A.P., The Lagrange problem, Izv. Akad. Nauk SSSR, MTT, 2, 1984.
7. BRATUS A.S. and SEIRANYAN A.P., Bimodal solutions in problems of optimizing eigenvalues, PMM, 47, 4, 1983.
8. BRATUS A.S. and SEIRANYAN A.P., The sufficient conditions for an extremum in problems of optimizing eigenvalues, PMM, 48, 4, 1984.
9. COURANT R. and HILBERT D., Methods of Mathematical Physics, 1, Moscow-Leningrad, Gostekhteorizdat, 1933.
10. KORN G. and KORN T., Handbook of Mathematics, Moscow, Nauka, 1970.
11. MASUR E.F., Optimal structural design under multiple eigenvalue constraints. Intern. J. Solids and Struct., 20, 3, 1984.
12. BRATUS A.S., Multiple eigenvalues in problems of optimizing the spectral properties of systems with a finite number of degrees of freedom, Zh. Vychisl. Mat. Mat. Fiz., 26, 5, 1986.
13. DEM'YANOV V.F. and MALOZEMOV V.N., Introduction to Minimax, Moscow, Nauka, 1972.

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